Parallel Joinable B-Tree in Fork-Join I/O Model

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Parallel Computational Models

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  - Later performs a *join* operation for sync

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- Span: Longest path within the graph

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  - Each fork and join spawns and syncs at most B threads

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  - Eventually develop *Multi-way Join*

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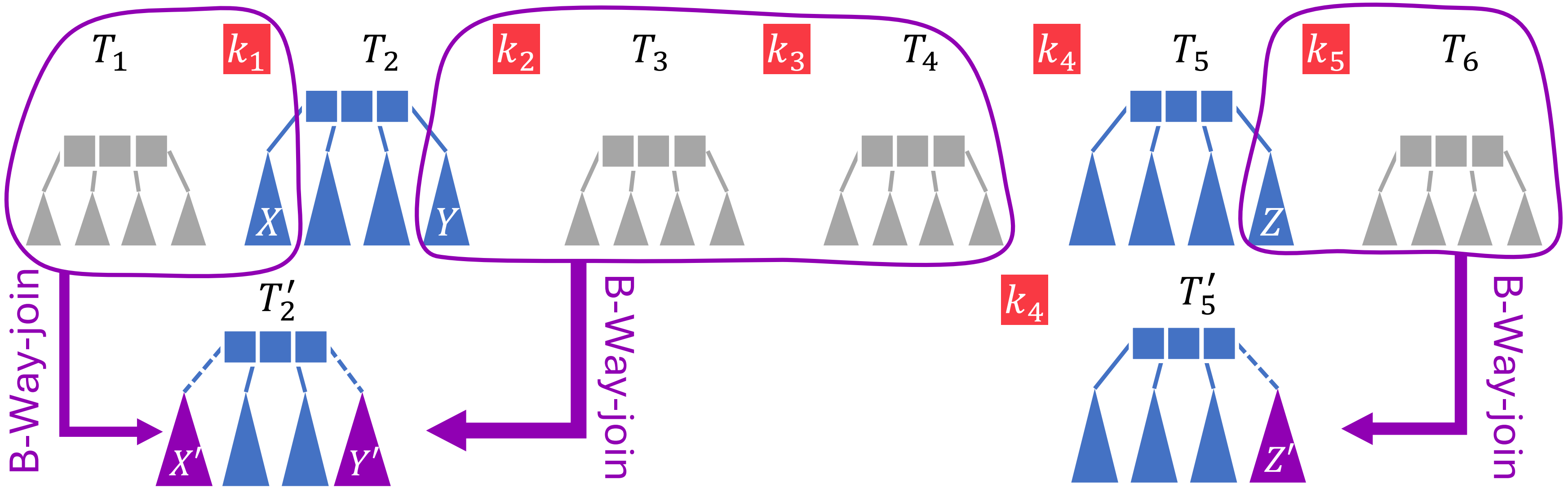
B-way Join

B-Way-join on five keys and six B-trees. $h(T_2) = h(T_5) = h^* = \max_i h(T_i)$, $B = 6$.

Input

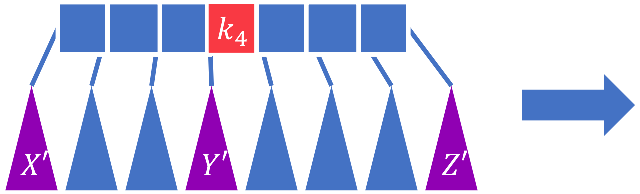
Step 1
Group inputs by the tall trees

Step 2
Recursively call Multi-join. Replace the left/right-most child of the tall trees



The purple trees have height $h^* - 1$ or height h^* (with one key at root). Note that T_2' and T_5' may not be strictly balanced at this point (X', Y' and Z' may be 1 level taller than other subtrees).

Step 3
Directly concatenate all (new) tall trees to the resulting tree & rebalance, based on Lemma 1



Resulting in a valid B-tree T with height h^* or $h^* + 1$; if T has height $h^* + 1$, then the root has at most $|C|$ keys, where $C \subseteq \{X', Y', Z'\}$ contains all trees with height h^*.

B-way Join

Theorem 1

Let $T_1, T_2, \ldots, T_{b+1}$ be a set of B-trees, with the largest tree height $h_{\max}$ and the shortest tree height $h_{\min}$, and $k_1, k_2, \ldots, k_b$ be a set of separator keys, where $b \leq B$. The *join* operation can be performed in $O(h_{\max} - h_{\min})$ I/O span and $O(b \cdot (h_{\max} - h_{\min}))$ I/O work.

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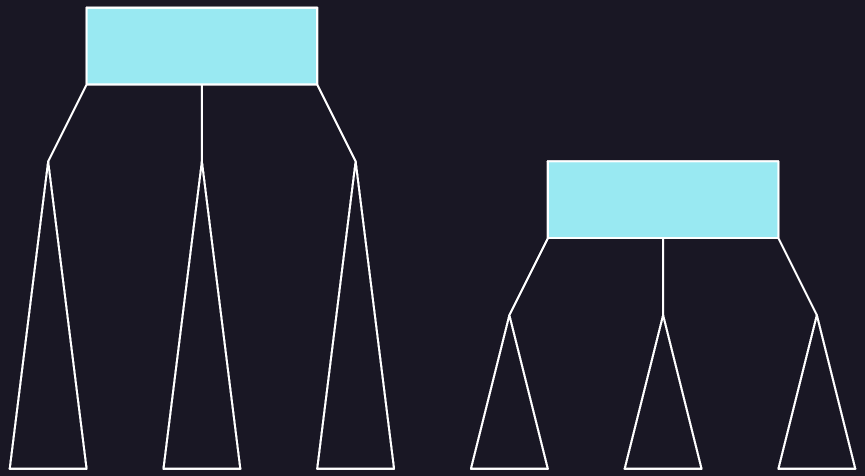
- Leads to a span-inefficient solution
- Union with above primitive leads to I/O Span $O(\log_B^2 n)$

B-way Join Improved

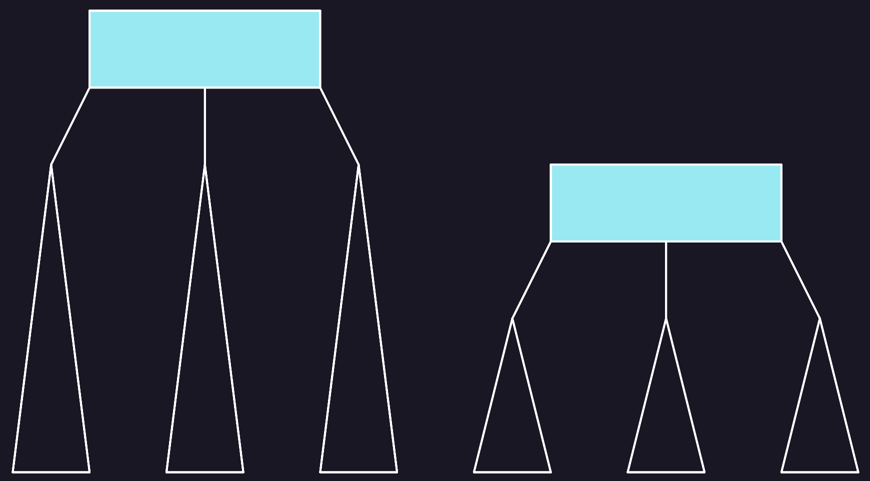
- *Grouping*
- Fuse within Groups
- Divide Large Nodes
- Divide Smaller Nodes
- *Concatenate and Rebalance*

Fuse within Groups



Fuse within Groups

- Do not recurse into subgroups of tall trees



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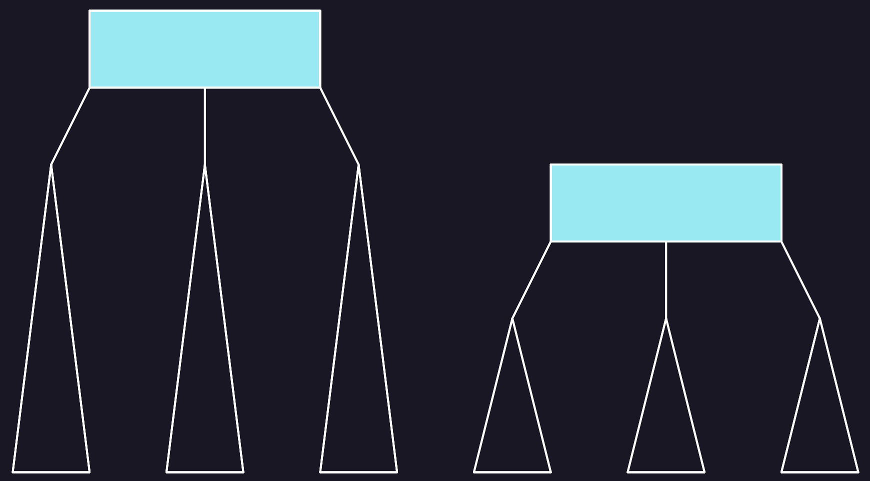

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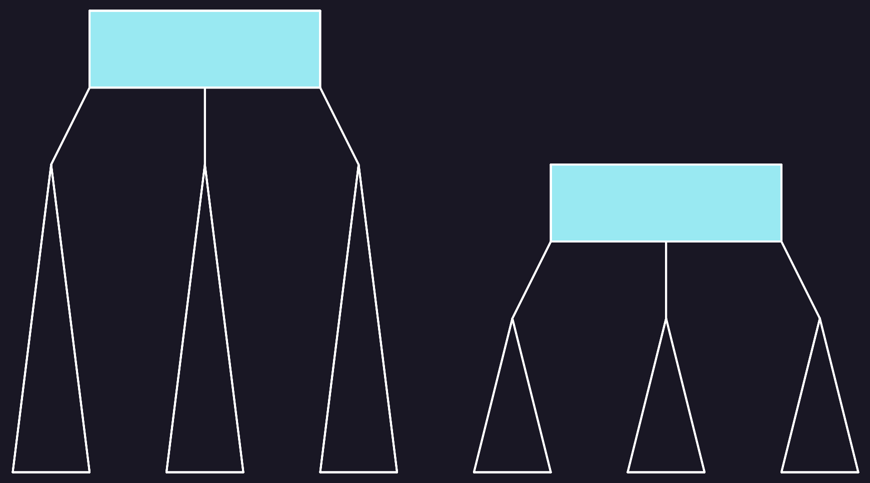

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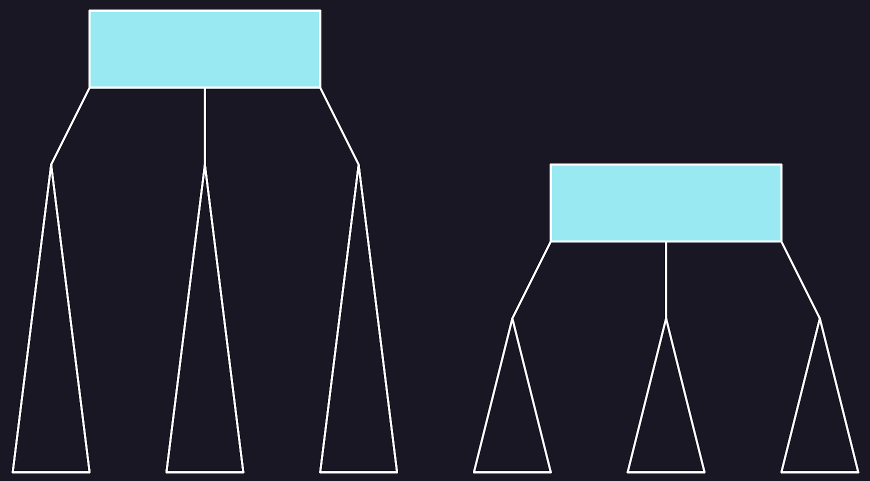
- Finding the node a root merges with takes $O(\log_B \log_B n)$ I/O span



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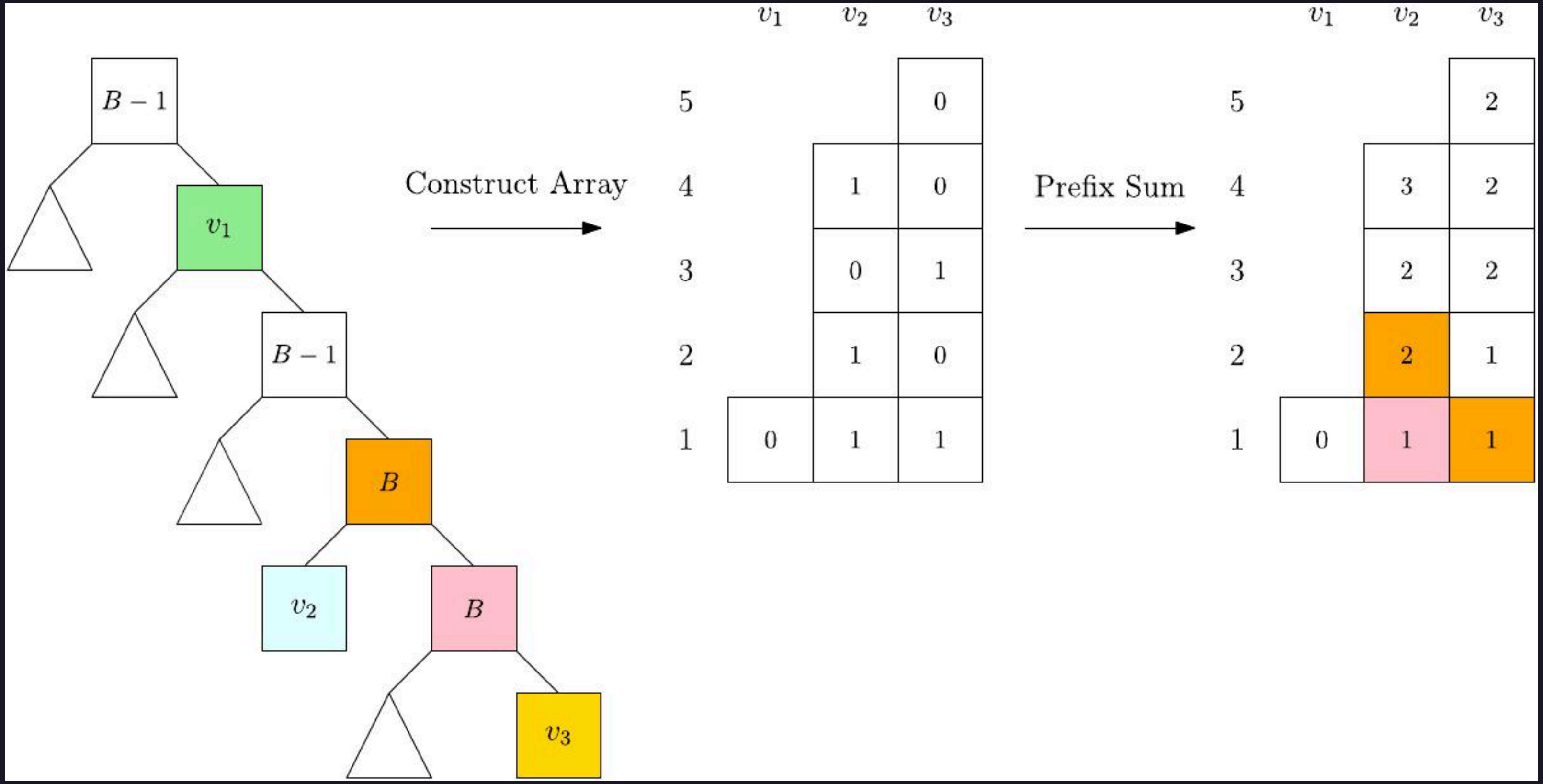
  - Each subsequent dividing of nodes will push up at most 1 key

Dividing Smaller Nodes



Multi-way Join I/O Span and Work

Theorem 2

We can join $T_1, \ldots, T_d$ B-trees and $k_1, \ldots, k_d$ keys together in parallel with $O(\log_B d \cdot \log_2 \log_B n + \log_B n)$ I/O span and $O(d \log_B n)$ I/O work, where n is $\sum_{i=1}^{d} |T_i|$, in the Fork-Join I/O model.

Multi-way Split I/O Span and Work

Theorem 3

We can split a B-tree T by $k_1, \ldots, k_d$ keys in parallel with $O(\log_B d + \log_B n)$ I/O span and $O(d \log_B n)$ I/O work, where n is $\sum_{i=1}^{d} |T_i|$, in the Fork-Join I/O Model.

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    - Minor modifications may be made to Multi-way Join to get Intersection and Difference operations

Union Operation

Theorem 4

Given two B-trees with sizes m and $n \geq m$, there exists a parallel algorithm that returns a new B-tree containing the union, intersection, and set difference of the two input trees in and $O\left(m \log_B \left(\frac{n}{m}\right)\right)$ I/O work, $O(\log_B m \cdot \log_2 \log_B n + \log_B n)$ I/O span, where B is the block size.

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- Apply algorithm to adaptive sorting settings
- Use Fork-Join I/O model on other algorithms

Thank you!